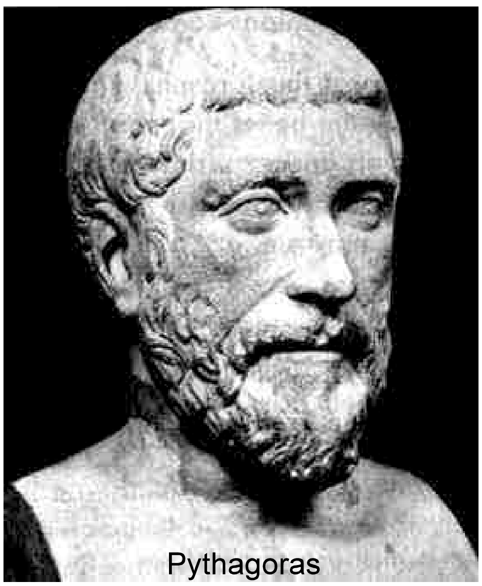
Pythagoras and the Pythagoreans[1]

Historically, the name **Pythagoras** means much more than the familiar namesake of the famous theorem about right triangles. The philosophy of Pythagoras and his school has become a part of the very fiber of mathematics, physics, and even the western tradition of liberal education, no matter what the discipline.



The stamp above depicts a coin issued by Greece on August 20, 1955, to commemorate the 2500th anniversary of the founding of the first school of philosophy by Pythagoras. Pythagorean philosophy was the prime source of inspiration for Plato and Aristotle whose influence on western thought is without question and is immeasurable.

1 Pythagoras and the Pythagoreans

Of his life, little is known. Pythagoras (fl 580-500, BC) was born in Samos on the western coast of what is now Turkey. He was reportedly the son of a substantial citizen, Mnesarchos. He met Thales, likely as a young man, who recommended he travel to Egypt. It seems certain that he gained much of his knowledge from the Egyptians, as had Thales before him. He had a reputation of having a wide range of knowledge over many subjects, though to one author as having little wisdom (Heraclitus) and to another as profoundly wise (Empedocles). Like Thales, there are no extant written works by Pythagoras or the Pythagoreans. Our knowledge about the Pythagoreans comes from others, including Aristotle, Theon of Smyrna, Plato, Herodotus, Philolaus of Tarentum, and others.

Pythagoras lived on Samos for many years under the rule of the tyrant Polycrates, who had a tendency to switch alliances in times of conflict — which were frequent. Probably because of continual conflicts and strife in Samos, he settled in Croton, on the eastern coast of Italy, a place of relative peace and safety. Even so, just as he arrived in about 532 BCE, Croton lost a war to neighboring city Locri, but soon thereafter defeated utterly the luxurious city of Sybaris. This is where Pythagoras began his society.

2 The Pythagorean School

The school of Pythagoras was every bit as much a religion as a school of mathematics. A rule of secrecy bound the members to the school, and oral communication was the rule. The Pythagoreans had numerous rules for everyday living. For example, here are a few of them:

- To abstain from beans.
- Not to pick up what has fallen.
- Not to touch a white cock.
- Not to stir the fire with iron.

⋮

- Do not look in a mirror beside a light.

Vegetarianism was strictly practiced probably because Pythagoras preached the transmigration of souls[2].

What is remarkable is that despite the lasting contributions of the Pythagoreans to philosophy and mathematics, the school of Pythagoras represents the **mystic tradition** in contrast with the scientific. Indeed, Pythagoras regarded himself as a mystic and even *semi-divine*. Said Pythagoras,

> "There are men, gods, and men like Pythagoras."

It is likely that Pythagoras was a charismatic, as well.

Life in the Pythagorean society was more-or-less egalitarian.

- The Pythagorean school regarded men and women equally.

[2] reincarnation

- They enjoyed a common way of life.
- Property was communal.
- Even mathematical discoveries were communal and by association attributed to Pythagoras himself — even from the grave. Hence, exactly what Pythagoras personally discovered is difficult to ascertain. Even Aristotle and those of his time were unable to attribute direct contributions from Pythagoras, always referring to 'the Pythagoreans', or even the 'so-called Pythagoreans'. Aristotle, in fact, wrote the book *On the Pythagoreans* which is now lost.

The Pythagorean Philosophy

The basis of the Pythagorean philosophy is simply stated:

> "There are three kinds of men and three sorts of people that attend the Olympic Games. The lowest class is made up of those who come to buy and sell, the next above them are those who compete. Best of all, however, are those who come simply to look on. The greatest purification of all is, therefore, disinterested science, and it is the man who devotes himself to that, the true philosopher, who has most effectually released himself from the 'wheel of birth'."[3]

The message of this passage is radically in conflict with modern values. We need only consider sports and politics.

⋆ Is not reverence these days is bestowed only on the "superstars"?

⋆ Are not there ubiquitous demands for *accountability*.

The *gentleman*[4], of this passage, has had a long run with this philosophy, because he was associated with the Greek genius, because

[3] Burnet, *Early Greek Philosophy*

[4] How many such philosophers are icons of the *western* tradition? We can include Hume, Locke, Descartes, Fermat, Milton, Göthe, Thoreau. Compare these names to Napoleon, Nelson, Bismark, Edison, Whitney, James Watt. You get a different feel.

the "virtue of contemplation" acquired theological endorsement, and because the ideal of disinterested truth dignified the academic life.

The Pythagorean Philosophy ála Bertrand Russell

From Bertrand Russell,[5], we have

> "It is to this gentleman that we owe pure mathematics. The contemplative ideal — since it led to pure mathematics — was the source of a useful activity. This increased it's prestige and gave it a success in theology, in ethics, and in philosophy."

Mathematics, so honored, became the model for other sciences. *Thought* became superior to the senses; *intuition* became superior to observation. The combination of mathematics and theology began with Pythagoras. It characterized the religious philosophy in Greece, in the Middle ages, and down through Kant. In Plato, Aquinas, Descartes, Spinoza and Kant there is a blending of religion and reason, of moral aspiration with logical admiration of what is timeless.

Platonism was essentially Pythagoreanism. The whole concept of an eternal world revealed to intellect but not to the senses can be attributed from the teachings of Pythagoras.

The Pythagorean School gained considerable influence in Croton and became politically active — on the side of the aristocracy. Probably because of this, after a time the citizens turned against him and his followers, burning his house. Forced out, he moved to **Metapontum**, also in Southern Italy. Here he died at the age of eighty. His school lived on, alternating between decline and re-emergence, for several hundred years. Tradition holds that Pythagoras left no written works, but that his ideas were carried on by eager disciples.

[5] *A History of Western Philosophy.* Russell was a logician, mathematician and philosopher from the first half of the twentieth century. He is known for attempting to bring pure mathematics into the scope of symbolic logic and for discovering some profound paradoxes in set theory.

3 Pythagorean Mathematics

What is known of the Pythagorean school is substantially from a book written by the Pythagorean, **Philolaus** (fl. c. 475 BCE) of Tarentum. However, according to the 3rd-century-AD Greek historian Diogenes Laërtius, he was born at Croton. After the death of Pythagoras, dissension was prevalent in Italian cities, Philolaus may have fled first to Lucania and then to Thebes, in Greece. Later, upon returning to Italy, he may have been a teacher of the Greek thinker Archytas. From his book Plato learned the philosophy of Pythagoras.

The dictum of the Pythagorean school was

All is number

The origin of this model may have been in the study of the constellations, where each constellation possessed a certain number of stars and the geometrical figure which it forms. What this dictum meant was that all things of the universe had a numerical attribute that uniquely described them. Even stronger, it means that all things which can be known or even conceived have number. Stronger still, not only do all things possess numbers, but all things *are* numbers. As Aristotle observes, the Pythagoreans regarded that number is both the principle matter for things and for constituting their attributes and permanent states. There are of course logical problems, here. (Using a basis to describe the same basis is usually a risky venture.) That Pythagoras could accomplish this came in part from further discoveries such musical harmonics and knowledge about what are now called Pythagorean triples. This is somewhat different from the Ionian school, where the elemental force of nature was some physical quantity such as water or air. Here, we see a model of the universe with number as its base, a rather abstract philosophy.

Even qualities, states, and other aspects of nature had descriptive numbers. For example,

- The number **one** : the number of reason.
- The number **two**: the first even or female number, the number of opinion.
- The number **three**: the first true male number, the number of harmony.

- The number **four**: the number of justice or retribution.
- The number **five**: marriage.
- The number **six**: creation

⋮

- The number **ten**: the *tetractys*, the number of the universe.

The Pythagoreans expended great effort to form the numbers from a single number, the Unit, (i.e. one). They treated the unit, which is a point without position, as a point, and a point as a unit having position. The unit was not originally considered a number, because a measure is not the things measured, but the measure of the One is the beginning of number.[6] This view is reflected in Euclid[7] where he refers to the multitude as being comprised of units, and a unit is that by virtue of which each of existing things is called one. The first definition of number is attributed to Thales, who defined it as a collection of units, clearly a derivate based on Egyptians arithmetic which was essentially grouping. Numerous attempts were made throughout Greek history to determine the root of numbers possessing some consistent and satisfying philosophical basis. This argument could certainly qualify as one of the earliest forms of the philosophy of mathematics.

The greatest of the numbers, ten, was so named for several reasons. Certainly, it is the base of Egyptian and Greek counting. It also contains the ratios of musical harmonies: 2:1 for the octave, 3:2 for the fifth, and 4:3 for the fourth. We may also note the only regular figures known at that time were the equilateral triangle, square, and pentagon[8] were also contained by within *tetractys*. Speusippus (d. 339 BCE) notes the geometrical connection.

Dimension:

One point: generator of dimensions (point).

Two points: generator of a line of dimension one

[6] Aristotle, *Metaphysics*

[7] *The Elements*

[8] Others such as the hexagon, octagon, etc. are easily constructed regular polygons with number of sides as multiples of these. The 15-*gon*, which is a multiple of three and five sides is also constructible. These polygons and their side multiples by powers of two were all those known.

Three points: generator of a triangle of dimension two

Four points: generator of a tetrahedron, of dimension three.

The sum of these is ten and represents all dimensions. Note the abstraction of concept. This is quite an intellectual distance from "fingers and toes".

Classification of numbers. The distinction between even and odd numbers certainly dates to Pythagoras. From Philolaus, we learn that

> "...number is of two special kinds, odd and even, with a third, even-odd, arising from a mixture of the two; and of each kind there are many forms."

And these, even and odd, correspond to the usual definitions, though expressed in unusual way[9]. But *even-odd* means a product of two and odd number, though later it is an even time an odd number. Other subdivisions of even numbers[10] are reported by Nicomachus (a neo-Pythagorean$\sim$100 A.D.).

- **even-even** — 2^n
- **even-odd** — $2(2m+1)$
- **odd-even** — $2^{n+1}(2m+1)$

Originally (our) number 2, the dyad, was not considered even, though Aristotle refers to it as the only even prime. This particular direction of mathematics, though it is based upon the earliest ideas of factoring, was eventually abandoned as not useful, though even and odd numbers and especially prime numbers play a major role in modern number theory.

Prime or **incomposite** numbers and **secondary** or **composite** numbers are defined in Philolaus:

[9]Nicomachus of Gerase (fl 100 CE) gives as ancient the definition that an *even* number is that which can be divided in to two equal parts and into two unequal part (except two), but however divided the parts must be of the same type (i.e. both even or both odd).

[10]Bear in mind that there is no zero extant at this time. Note, the "experimentation" with definition. The same goes on today. Definitions and directions of approach are in a continual flux, then and now.

- A **prime** number is rectilinear, meaning that it can only be set out in one dimension. The number 2 was not originally regarded as a prime number, or even as a number at all.
- A **composite** number is that which is measured by (has a factor) some number. (Euclid)
- Two numbers are **prime to one another** or **composite to one another** if their greatest common divisor[11] is one or greater than one, respectively. Again, as with even and odd numbers there were numerous alternative classifications, which also failed to survive as viable concepts.[12]

For prime numbers, we have from Euclid the following theorem, whose proof is considered by many mathematicians as the quintessentially most elegant of all mathematical proofs.

Proposition. There are an infinite number of primes.

Proof. (Euclid) Suppose that there exist only finitely many primes $p_1 < p_2 < ... < p_r$. Let $N = (p_1)(p_2)...(p_r) > 2$. The integer $N - 1$, being a product of primes, has a prime divisor p_i in common with N; so, p_i divides $N - (N - 1) = 1$, which is absurd!

The search for primes goes on. **Eratsothenes** (276 B.C. - 197 B.C.)[13], who worked in Alexandria, devised a *sieve* for determining primes. This sieve is based on a simple concept:

Lay off all the numbers, then mark of all the multiples of 2, then 3, then 5, and so on. A prime is determined when a number is not marked out. So, 3 is uncovered after the multiples of two are marked out; 5 is uncovered after the multiples of two and three are marked out. Although it is not possible to determine large primes in this fashion, the sieve was used to determine early tables of primes. (This makes a wonderful exercise in the discovery of primes for young students.)

[11] in modern terms

[12] We have

- **prime and incomposite** — ordinary primes excluding 2,
- **secondary and composite** — ordinary composite with prime factors only,
- **relatively prime** — two composite numbers but prime and incomposite to another number, e.g. 9 and 25. Actually the third category is wholly subsumed by the second.

[13] Eratsothenes will be studied in somewhat more detail later, was gifted in almost every intellectual endeavor. His admirers call him the second Plato and some called him *beta*, indicating that he was the second of the wise men of antiquity.

It is known that there is an infinite number of primes, but there is no way to find them. For example, it was only at the end of the 19th century that results were obtained that describe the asymptotic density of the primes among the integers. They are relatively sparce as the following formula

$$\text{The number of primes} \leq n \sim \frac{n}{\ln n}$$

shows.[14] Called the Prime Number Theorem, this celebrated results was not even conjectured in its correct form until the late 18th century and its proof uses mathematical machinery well beyond the scope of the entirety of ancient Greek mathematical knowledge. The history of this theorem is interesting in its own right and we will consider it in a later chapter. For now we continue with the Pythagorean story.

The pair of numbers a and b are called **amicable** or **friendly** if the divisors of a sum to b and if the divisors of b sum to a. The pair 220 and 284, were known to the Greeks. Iamblichus (C.300 -C.350 CE) attributes this discovery to Pythagoras by way of the anecdote of Pythagoras upon being asked 'what is a friend' answered '*Alter ego*', and on this thought applied the term directly to numbers pairs such as 220 and 284. Among other things it is not known if there is infinite set of amicable pairs. Example: All primes are deficient. More interesting that amicable numbers are perfect numbers, those numbers amicable to themselves. Mathematically, a number n is **perfect** if the sum of its divisors is itself.

Examples: (6, 28, 496, 8128, ...)

$$\begin{aligned} 6 &= 1+2+3 \\ 28 &= 1+2+4+7+14 \\ 496 &= 1+2+4+8+16+31+62+124+248 \end{aligned}$$

There are no direct references to the Pythagorean study of these numbers, but in the comments on the Pythagorean study of amicable numbers, they were almost certainly studied as well. In Euclid, we find the following proposition.

Theorem. (Euclid) If $2^p - 1$ is prime, then $(2^p - 1)2^{p-1}$ is perfect.
Proof. The proof is straight forward. Suppose $2^p - 1$ is prime. We identify all the factors of $(2^p - 1)2^{p-1}$. They are

[14] This asymptotic result if also expressed as follows. Let $P(n) =$ The number of primes $\leq n$. Then $\lim_{n\to\infty} P(n)/[\frac{n}{\ln n}] = 1$.

$$1,\ 2,\ 4,\ \ldots,\ 2^{p-1}, \quad \text{and}$$
$$1\cdot(2^{p-1}-1),\ 2\cdot(2^{p-1}-1),\ 4\cdot(2^{p-1}-1),\ ,\ldots,\ 2^{p-2}\cdot(2^{p-1}-1)$$

Adding we have[15]

$$\begin{aligned}\sum_{n=0}^{p-1} 2^n + (2^{p-1}-1)\sum_{n=0}^{p-2} 2^n &= 2^p - 1 + (2^p-1)(2^{p-1}-1) \\ &= (2^p-1)2^{p-1}\end{aligned}$$

and the proof is complete.

(Try, $p = 2, 3, 5$, and 7 to get the numbers above.) There is just something about the word "perfect". The search for perfect numbers continues to this day. By Euclid's theorem, this means the search is for primes of the form $(2^p - 1)$, where p is a prime. The story of and search for perfect numbers is far from over. First of all, it is not known if there are an infinite number of perfect numbers. However, as we shall soon see, this hasn't been for a lack of trying. Completing this concept of describing of numbers according to the sum of their divisors, the number a is classified as **abundant** or **deficient**[16] according as their divisors sums greater or less than a, respectively. Example: The divisors of 12 are: 6,4,3,2,1 — Their sum is 16. So, 12 is abundant. Clearly all prime numbers, with only one divisor (namely, 1) are deficient.

In about 1736, one of history's greatest mathematicians, Leonhard Euler (1707 - 1783) showed that all even perfect numbers must have the form given in Euclid's theorem. This theorem stated below is singularly remarkable in that the individual contributions span more than two millenia. Even more remarkable is that Euler's proof could have been discovered with known methods from the time of Euclid. The proof below is particularly elementary.

Theorem (Euclid - Euler) An even number is perfect if and only if it has the form $(2^p - 1)2^{p-1}$ where $2^p - 1$ is prime.
Proof. The sufficiency has been already proved. We turn to the necessity. The slight change that Euler brings to the description of perfect numbers is that he includes the number itself as a divisor. Thus a perfect is one whose divisors add to twice the number. We use this new definition below. Suppose that m is an even perfect number. Factor m

[15]Recall, the geometric series $\sum_{n=0}^{N} r^n = \frac{r^{N+1}-1}{r-1}$. This was also well known in antiquity and is in Euclid, *The Elements*.
[16]Other terms used were *over-perfect* and *defective* respectively for these concepts.

as $2^{p-1}a$, where a is odd and of course $p > 1$. First, recall that the sum of the factors of 2^{p-1}, when 2^{p-1} itself is included, is $(2^p - 1)$ Then

$$2m = 2^p a = (2^p - 1)(a + \cdots + 1)$$

where the term $\cdots$ refers to the sum of all the other factors of a. Since $(2^p - 1)$ is odd and 2^p is even, it follows that $(2^p - 1)|a$, or $a = b(2^p - 1)$. First assume $b > 1$. Substituting above we have $2^p a = 2^p(2^p - 1)b$ and thus

$$\begin{aligned} 2^p(2^p - 1)b &= (2^p - 1)((2^p - 1)b + (2^p - 1) + b + \cdots + 1) \\ &= (2^p - 1)(2^p + 2^p\, b + \cdots) \end{aligned}$$

where the term $\cdots$ refers to the sum of all other the factors of a. Cancel the terms $(2^p - 1)$. There results the equation

$$2^p b = 2^p + 2^p\, b + \cdots$$

which is impossible. Thus $b = 1$. To show that $(2^p - 1)$ is prime, we write a similar equation as above

$$\begin{aligned} 2^p(2^p - 1) &= (2^p - 1)((2^p - 1) + \cdots + 1) \\ &= (2^p - 1)(2^p + \cdots) \end{aligned}$$

where the term $\cdots$ refers to the sum of all other the factors of $(2^p - 1)$. Now cancel $(2^p - 1)$. This gives

$$2^p = (2^p + \cdots)$$

If there are any other factors of $(2^p - 1)$, this equation is impossible. Thus, $(2^p - 1)$ is prime, and the proof is complete. ■

4 The Primal Challenge

The search for large primes goes on. Prime numbers are so fundamental and so interesting that mathematicians, amateur and professional, have been studying their properties ever since. Of course, to determine if a given number n is prime, it is necessary only to check for divisibility by a prime up to $\sqrt{n}$. (Why?) However, finding large primes in this way is nonetheless impractical[17] In this short section, we depart history and

[17] The current record for largest prime has more than a million digits. The square root of any test prime then has more than 500,000 digits. Testing a million digit number against all such primes less than this is certainly impossible.

take a short detour to detail some of the modern methods employed in the search. Though this is a departure from ancient Greek mathematics, the contrast and similarity between then and now is remarkable. Just the fact of finding perfect numbers using the previous propositions has spawned a cottage industry of determining those numbers p for which $2^p - 1$ is prime. We call a prime number a **Mersenne Prime** if it has the form $2^p - 1$ for some positive integer p. Named after the friar **Marin Mersenne** (1588 - 1648), an active mathematician and contemporary of Fermat, Mersenne primes are among the largest primes known today. So far 38 have been found, though it is unknown if there are others between the 36th and 38th. It is not known if there are an infinity of Mersenne primes. From Euclid's theorem above, we also know exactly 38 perfect numbers. It is relatively routine to show that if $2^p - 1$ is prime, then so also is p.[18] Thus the known primes, say to more than ten digits, can be used to search for primes of millions of digits.

Below you will find complete list of Mersenne primes as of January, 1998. A special method, called the *Lucas-Lehmer* test has been developed to check the primality the Mersenne numbers.

[18] If $p = rs$, then $2^p - 1 = 2^{rs} - 1 = (2^r)^s - 1 = (2^r - 1)((2^r)^{s-1} + (2^r)^{s-2} \cdots + 1)$

Number	**Prime** (exponent)	**Digits**	**Mp**	**Year**	**Discoverer**
1	2	1	1	—	Ancient
2	3	1	2	—	Ancient
3	5	2	3	—	Ancient
4	7	3	4	—	Ancient
5	13	4	8	1456	anonymous
6	17	6	10	1588	Cataldi
7	19	6	12	1588	Cataldi
8	31	10	19	1772	Euler
9	61	19	37	1883	Pervushin
10	89	27	54	1911	Powers
11	107	33	65	1914	Powers
12	127	39	77	1876	Lucas
13	521	157	314	1952	Robinson
14	607	183	366	1952	Robinson
15	1279	386	770	1952	Robinson
16	2203	664	1327	1952	Robinson
17	2281	687	1373	1952	Robinson
18	3217	969	1937	1957	Riesel
19	4253	1281	2561	1961	Hurwitz
20	4423	1332	2663	1961	Hurwitz
21	9689	2917	5834	1963	Gillies
22	9941	2993	5985	1963	Gillies
23	11213	3376	6751	1963	Gillies
24	19937	6002	12003	1971	Tuckerman
25	21701	6533	13066	1978	Noll - Nickel
26	23209	6987	13973	1979	Noll
27	44497	13395	26790	1979	Nelson - Slowinski
28	86243	25962	51924	1982	Slowinski
29	110503	33265	66530	1988	Colquitt - Welsh
30	132049	39751	79502	1983	Slowinski
31	216091	65050	130100	1985	Slowinski
32	756839	227832	455663	1992	Slowinski & Gage
33	859433	258716	517430	1994	Slowinski & Gage
34	1257787	378632	757263	1996	Slowinski & Gage
35	1398269	420921	841842	1996	Armengaud, Woltman,
??	2976221	895932	1791864	1997	Spence, Woltman,
??	3021377	909526	1819050	1998	Clarkson, Woltman, Kurowski
??	26972593	2098960		1999	Hajratwala, Kurowski
??	213466917	4053946		2001	Cameron, Kurowski

What about odd perfect numbers? As we have seen Euler characterized all even perfect numbers. But nothing is known about odd perfect numbers except these few facts:

- If n is an odd perfect number, then it must have the form

  $$n = q^2 \cdot p^{2k+1},$$

  where p is prime, q is an odd integer and k is a nonnegative integer.
- It has at least 8 different prime factors and at least 29 prime factors.
- It has at least 300 decimal digits.

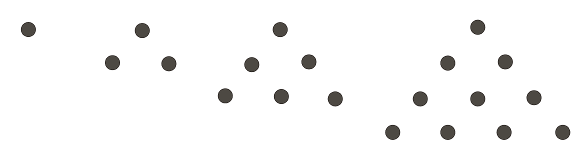
Truly a challenge, finding an odd perfect number, or proving there are none will resolve the one of the last open problems considered by the Greeks.

5 Figurate Numbers.

Numbers geometrically constructed had a particular importance to the Pythagoreans.

Triangular numbers. These numbers are 1, 3, 6, 10, ... The general form is the familiar

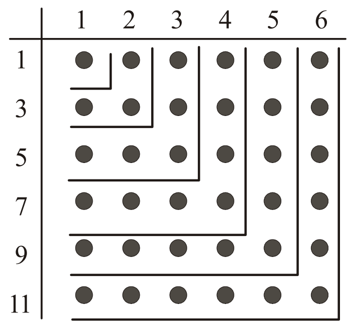
$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$



Triangular Numbers

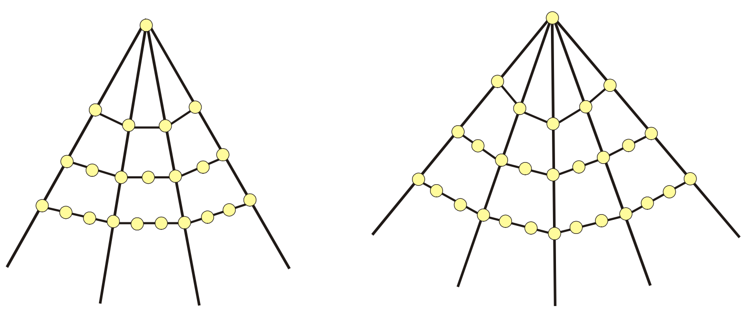
Square numbers These numbers are clearly the squares of the integers 1, 4, 9, 16, and so on. Represented by a square of dots, they prove(?) the well known formula

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2.$$



Square Numbers

The *gnomon* is basically an architect's template that marks off "similar" shapes. Originally introduced to Greece by Anaximander, it was a Babylonian astronomical instrument for the measurement of time. It was made of an upright stick which cast shadows on a plane or hemispherical surface. It was also used as an instrument to measure right angles, like a modern carpenter's square. Note the *gnomon* has been placed so that at each step, the next odd number of dots is placed. The **pentagonal** and **hexagonal** numbers are shown in the below.


Pentagonal Numbers Hexagonal Numbers

Figurate Numbers of any kind can be calculated. Note that the sequences have sums given by

$$1 + 4 + 7 + \ldots + (3n - 2) = \frac{3}{2}n^2 - \frac{1}{2}n$$

and

$$1 + 5 + 9 + \ldots + (4n - 3) = 2n^2 - n.$$

Similarly, polygonal numbers of all orders are designated; this process can be extended to three dimensional space, where there results the **polyhedral numbers**. Philolaus is reported to have said:

> *All things which can be known have number; for it is not possible that without number anything can be either conceived or known.*

6 Pythagorean Geometry

6.1 Pythagorean Triples and The Pythagorean Theorem

Whether Pythagoras learned about the 3, 4, 5 right triangle while he studied in Egypt or not, he was certainly aware of it. This fact though could not but strengthen his conviction that *all is number*. It would also have led to his attempt to find other forms, i.e. triples. How might he have done this?

One place to start would be with the square numbers, and arrange that three consecutive numbers be a Pythagorean triple! Consider for any odd number m,

$$m^2 + (\frac{m^2 - 1}{2})^2 = (\frac{m^2 + 1}{2})^2$$

which is the same as

$$m^2 + \frac{m^4}{4} - \frac{m^2}{2} + \frac{1}{4} = \frac{m^4}{4} + \frac{m^2}{2} + \frac{1}{4}$$

or

$$m^2 = m^2$$

Now use the gnomon. Begin by placing the gnomon around n^2. The next number is $2n+1$, which we suppose to be a square.

$$2n+1=m^2,$$

which implies

$$n=\frac{1}{2}(m^2-1),$$

and therefore

$$n+1=\frac{1}{2}(m^2+1).$$

It follows that

$$m^2+\frac{m^4}{4}-\frac{m^2}{2}+\frac{1}{4}=\frac{m^4}{4}+\frac{m^2}{2}+\frac{1}{4}$$

This idea evolved over the years and took other forms. The essential fact is that the Pythagoreans were clearly aware of the Pythagorean theorem

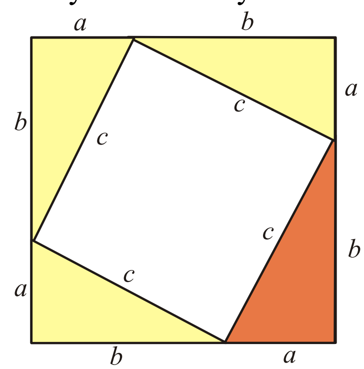
Did Pythagoras or the Pythagoreans actually prove the Pythagorean theorem? (See the statement below.) Later writers that attribute the proof to him add the tale that he sacrificed an ox to celebrate the discovery. Yet, it may have been Pythagoras's religious mysticism may have prevented such an act. What is certain is that Pythagorean triples were known a millennium before Pythagoras lived, and it is likely that the Egyptian, Babylonian, Chinese, and India cultures all had some "proto-proof", i.e. justification, for its truth. The proof question remains.

No doubt, the earliest "proofs" were arguments that would not satisfy the level of rigor of later times. Proofs were refined and retuned repeatedly until the current form was achieved. Mathematics is full of arguments of various theorems that satisfied the rigor of the day and were later replaced by more and more rigorous versions.[19] However, probably the Pythagoreans attempted to give a proof which was up to the rigor of the time. Since the Pythagoreans valued the idea of proportion, it is plausible that the Pythagoreans gave a proof based on proportion similar to Euclid's proof of Theorem 31 in Book VI of *The Elements*. The late Pythagoreans (~400 BCE) however probably did supply a rigorous proof of this most famous of theorems.

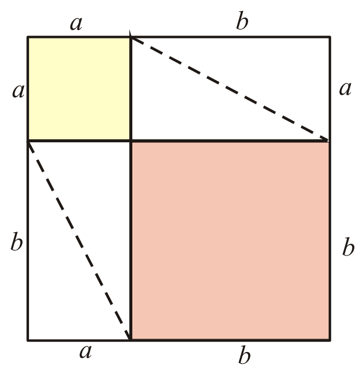
[19]One of the most striking examples of this is the Fundamental Theorem of Algebra, which asserts the existence of at least one root to any polynomial. Many proofs, even one by Euler, passed the test of rigor at the time, but it was Carl Friedrich Gauss (1775 - 1855) that gave us the first proof that measures up to modern standards of rigor.

There are numerous proofs, more than 300 by one count, in the literature today, and some of them are easy to follow. We present three of them. The first is a simple appearing proof that establishes the theorem by visual diagram. To "rigorize" this theorem takes more than just the picture. It requires knowledge about the similarity of figures, and the Pythagoreans had only a limited theory of similarity.

$$\begin{aligned} (a+b)^2 &= c^2 + 4(\frac{1}{2}\,ab) \\ a^2 + 2ab + b^2 &= c^2 + 2ab \\ a^2 + b^2 &= c^2 \end{aligned}$$

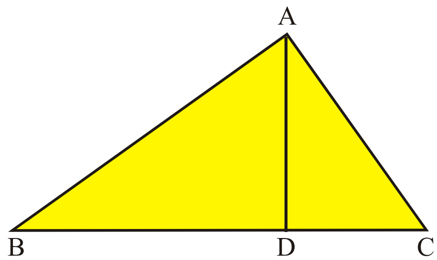


This proof is based upon Books I and II of Euclid's *Elements*, and is supposed to come from the figure to the right. Euclid allows the decomposition of the square into the two boxes and two rectangles. The rectangles are cut into the four triangles shown in the figure.



Then the triangle are reassembled into the first figure.

The next proof is based on similarity and proportion and is a special case of Theorem 31 in Book VI of *The Elements*. Consider the figure below.



If ABC is a right triangle, with right angle at A, and AD is perpendicular to BC, then the triangles DBA and DAC are similar to ABC. Applying the proportionality of sides we have

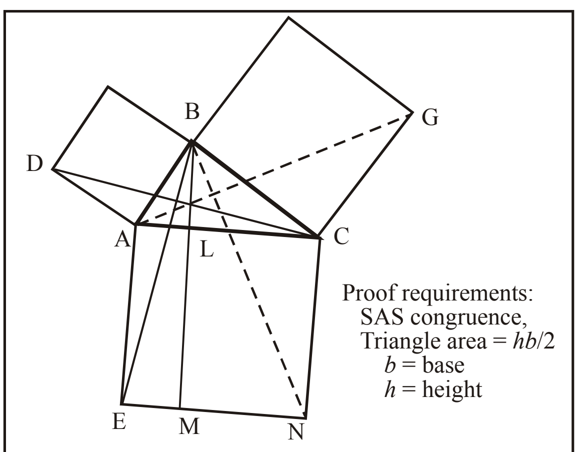
$$\begin{aligned} |BA|^2 &= |BD|\,|BC| \\ |AC|^2 &= |CD|\,|BC| \end{aligned}$$

It follows that

$$|BA|^2 + |AC|^2 = |BC|^2$$

Finally we state and prove what is now called the Pythagorean Theorem as it appears in Euclid *The Elements*.

Theorem I-47. In right-angled triangles, the square upon the hypotenuse is equal to the sum of the squares upon the legs.



Pythagorean Theorem

This diagram is identical to the original figure used in the Euclid's proof theorem. The figure was known to Islamic mathematicians as the *Figure of the Bride*.

Sketch of Proof. Note that triangles $\triangle ADC$ and $\triangle ADE$ are congruent and hence have equal area. Now slide the vertex C of $\triangle ADC$ to B. Slide also the vertex B of $\triangle ADE$ to L. Each of these transformations do not change the area. Therefore, by doubling, it follows that the area of the rectangle $ALME$ is equal to the area of the square upon the side AB. Use a similar argument to show that the area of the square upon the side BC equals the area of the rectangle $LCNM$. ■

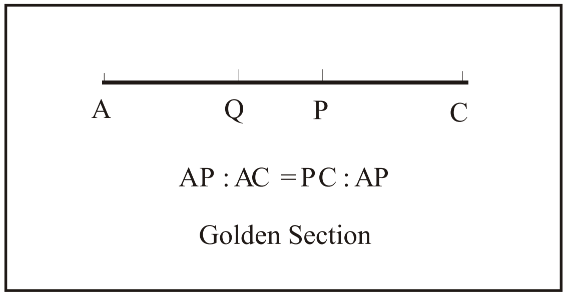
This stamp was issued by Greece. It depicts the Pythagorean theorem.

6.2 The Golden Section

From Kepler we have these words

> "Geometry has two great treasures: one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel."

A line AC divided into **extreme and mean ratio** is defined to mean that it is divided into two parts, AP and PC so that AP:AC=PC:AP, where AP is the longer part.



Let $AP = x$ and $AC = a$. Then the golden section is

$$\frac{x}{a} = \frac{a - x}{x},$$

and this gives the quadratic equation

$$x^2 + ax - a^2.$$

The solution is

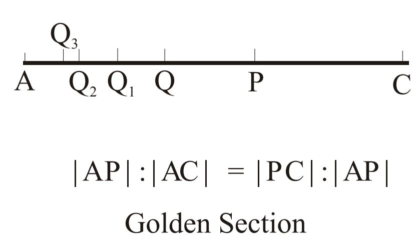
$$x = \frac{-1 \pm \sqrt{5}}{2}\, a.$$

The **golden section**[20] is the positive root:

$$x = \frac{\sqrt{5} - 1}{2} \sim .62$$

The point Q in the diagram above is positioned at a distance from A so that $|AQ| = |PC|$. As such the segment AP is divided into mean and extreme ratio by Q. Can you prove this? Of course, this idea can be applied recursively, to successive refinements of the segment all into such sections.

In the figure to the right $Q_1, Q_2, Q_3, \ldots$ are selected so that $|AQ_1| = |QP|$, $|AQ_2| = |Q_1Q|$,
$|AQ_3| = |Q_2Q_1|, \ldots$ respectively.



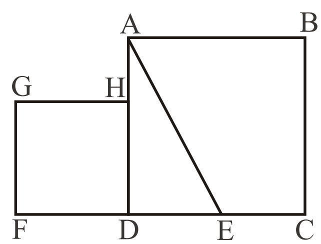
[20]...now called the *Golden ratio.* Curiously, this number has recurred throughout the development of mathematics. We will see it again and again.

The points $Q_1,\ Q_2,\ ,Q_3,\ldots$ divide the segments $AQ,\ AQ_1,| > AQ_2,\ldots$ into extreme and mean ratio, respectively.

The Pythagorean Pentagram

And this was all connected with the construction of a pentagon. First we need to construct the golden section. The geometric construction, the only kind accepted[21], is illustrated below.

Assume the square ABCE has side length a. Bisecting DC at E construct the diagonal AE, and extend the segment ED to EF, so that EF=AE. Construct the square DFGH. The line AHD is divided into extrema and mean ratio.



Golden Section

Verification:

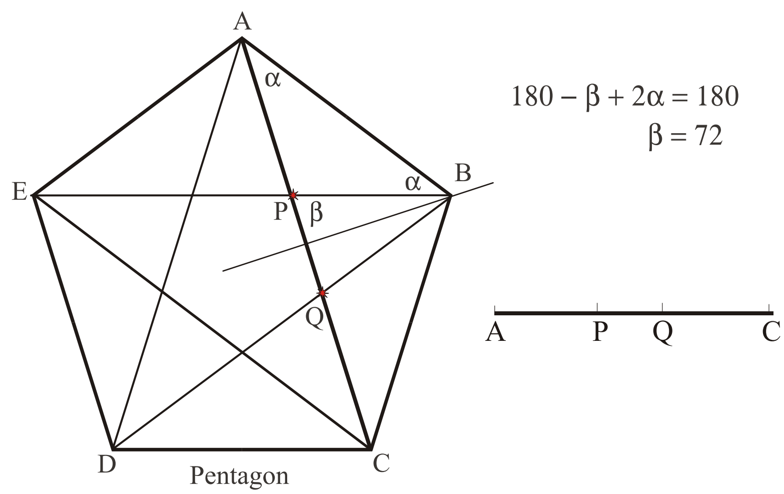
$$|AE|^2 = |AD|^2 + |DE|^2 = a^2 + (a/2)^2 = \frac{5}{4}a^2.$$

Thus,

$$|DH| = (\frac{\sqrt{5}}{2} - \frac{1}{2})a = \frac{\sqrt{5}-1}{2}\,a.$$

The key to the compass and ruler construction of the pentagon is the construction of the isosceles triangle with angles 36°, 72°, and 72°. We begin this construction from the line AC in the figure below.

[21]In actual fact, the Greek "fixation" on geometric methods to the exclusion of algebraic methods can be attributed to the influence of Eudoxus



Divide a line AC into the 'section' with respect to both endpoints. So PC:AC=AP:PC; also AQ:AC=QC:AQ. Draw an arc with center A and radius AQ. Also, draw an arc with center C with radius PC. Define B to be the intersection of these arcs. This makes the triangles AQB and CBP congruent. The triangles BPQ and AQB are similar, and therefore $PQ : QB = QP : AB$. Thus the angle $\angle PBQ = \angle QABAB = AQ$.

Define $\alpha := \angle$ PAB and $\beta := \angle$ QPB. Then $180^o - \beta - 2\alpha = 180^o$. This implies $\alpha = \frac{1}{2}\beta$, and hence $(2 + \frac{1}{2})\beta = 180$. Solving for β we, get $\beta = 72^o$. Since $\triangle$ PBQ is isoceles, the angle $\angle$ QBP $= 32^o$. Now complete the line BE=AC and the line BD=AC and connect edges AE, ED and DC. Apply similarity of triangles to show that all edges have the same length. This completes the proof.

6.3 Regular Polygons

The only regular polygons known to the Greeks were the equilaterial triangle and the pentagon. It was not until about 1800 that C. F. Guass added to the list of constructable regular polyons by showing that there are three more, of 17, 257, and 65,537 sides respectively. Precisely, he showed that the constructable regular polygons must have

$$2^m p_1 p_2 \dots p_r$$

sides where the $p_1, \ldots, p_r$ are distinct **Fermat primes**. A Fermat prime is a prime having the form

$$2^{2^n} + 1.$$

In about 1630, the Frenchman Pierre de Fermat (1601 - 1665) conjectured that all numbers of this kind are prime. But now we know differently.

Pierre Fermat (1601-1665), was a court attorney in Toulouse (France). He was an avid mathematician and even participated in the fashion of the day which was to reconstruct the masterpieces of Greek mathematics. He generally refused to publish, but communicated his results by letter.

Are there any other Fermat primes? Here is all that is known to date. It is not known if any other of the Fermat numbers are prime.

p	$2^{2^p}+1$	Factors	Discoverer
0	3	3	ancient
1	5	5	ancient
2	17	17	ancient
3	257	257	ancient
4	65537	65537	ancient
5	4,294,957,297	641, 6,700,417	Euler, 1732
6	21	274177,67280421310721	
7	39 digits	composite	
8	78 digits	composite	
9	617 digits	composite	Lenstra, et.al., 1990
10	709 digits	unknown	
11	1409 digits	composite	Brent and Morain, 1988
12-20		composite	

By the theorem of Gauss, there are constructions of regular polygons of only 3, 5 ,15 , 257, and 65537 sides, plus multiples,

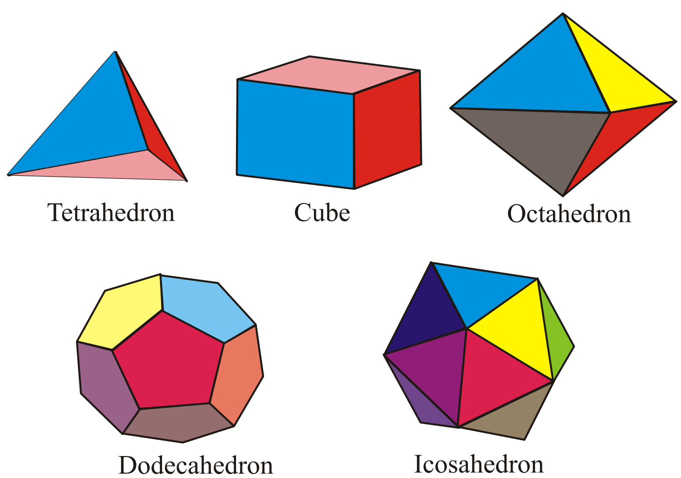
$$2^m p_1 p_2 \dots p_r$$

sides where the $p_1, \dots, p_r$ are distinct **Fermat primes**.

6.4 More Pythagorean Geometry

Contributions[22] by the Pythagoreans include

- Various theorems about triangles, parallel lines, polygons, circles, spheres and regular polyhedra. In fact, the sentence in Proclus about the discovery of the irrationals also attributes to Pythagoras the discovery of the five regular solids (called then the 'cosmic figures'). These solids, the tetrahedron (4 sides, triangles), cube (6 sides, squares, octahedron (8 sides, triangles), dodecahedron (12 sides, pentagons), and icosahedron (20 sides, hexagons) were possibly known to Pythagoras, but it is unlikely he or the Pythagoreans could give rigorous constructions of them. The first four were associated with the four elements, earth, fire, air, and water, and because of this they may not have been aware of the icosahedron. Usually, the name Theaetetus is associated with them as the mathematician who proved there are only five, and moreover, who gave rigorous constructions.



- Work on a class of problems in the applications of areas. (e.g. to construct a polygon of given area and similar to another polygon.)

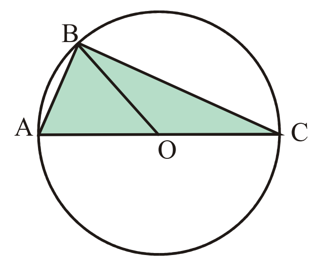
- The geometric solutions of quadratics. For example, given a line segment, construct on part of it or on the line segment extended a parallelogram equal to a given rectilinear figure in area and falling

[22] These facts generally assume a knowledge of the Pythagorean Theorem, as we know it. The level of rigor has not yet achieved what it would become by the time of Euclid

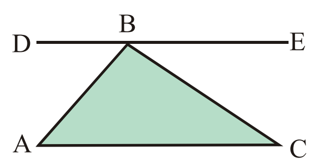
short or exceeding by a parallelogram similar to a given one. (In modern terms, solve $\frac{b}{c}x^2 + ax = d$.)

6.5 Other Pythagorean Geometry

We know from from Eudemus that the Pythagoreans discovered the result that the sum of the angles of any triangle is the sum of two right angles. However, if Thales really did prove that every triangle inscribed in a right triangle is a right triangle, he surely would have noted the result for right triangles. This follows directly from observing that the base angles of the isosceles traingles formed from the center as in the figure just to the right. The proof for any triangle follows directly. However, Eudemus notes a different proof. This proof requires the "alternating interior angles" theorem. That is:

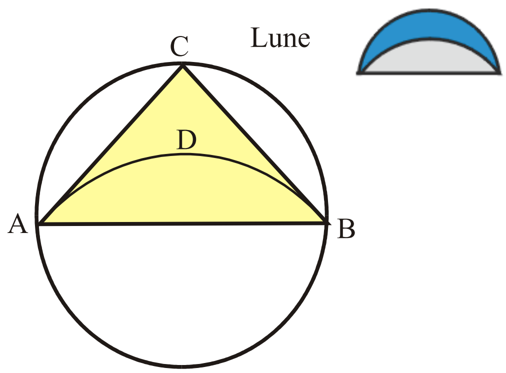


Theorem. (Euclid, *The Elements* Book I, Proposition 29.) *A straight line falling on parallel straight lines make the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles.*



From this result and the figure just above, note that the angles $\angle ABD = \angle CAB$ and $\angle CBE = \angle ACB$. The result follows. ■

The quadrature of certain **lunes** (crescent shaped regions) was performed by **Hippocrates of Chios**. He is also credited with the arrangement of theorems in an order so that one may be proved from a previous one (as we see in Euclid).



We wish to determine the area of the lune $ABCD$, where the large segment ABD is similar to the smaller segment (with base on one leg of the right isosceles triangle $\triangle ABC$). Because segments are to each other as the squares upon their bases, we have the

Proposition: The area of the large lune $ABCD$ is the area of the triangle $\triangle ABC$.

This proposition was among the first that determined the area of a curvilinear figure in terms of a rectilinear figure. Quadratures were obtained for other lunes, as well. There resulted great hope and encouragement that the circle could be squared. This was not to be.

7 The Pythagorean Theory of Proportion

Besides discovering the five regular solids, Pythagoras also discovered the theory of proportion. Pythagoras had probably learned in Babylon the three basic means, the **arithmetic**, the **geometric**, and the **subcontrary** (later to be called the **harmonic**).

Beginning with $a > b > c$ and denoting b as the —**mean** of a and c, they are:

$$1 \qquad \frac{a-b}{b-c} = \frac{a}{a} \qquad \textit{arithmetic} \quad a+c=2b$$

2	$\dfrac{a-b}{b-c} = \dfrac{a}{b}$	*geometric*	$ac = b^2$
3	$\dfrac{a-b}{b-c} = \dfrac{a}{c}$	*harmonic*	$\dfrac{1}{a} + \dfrac{1}{c} = \dfrac{2}{b}$

The most basic fact about these proportions or means is that if $a > c$, then $a > b > c$. In fact, Pythagoras or more probably the Pythagoreans added seven more proportions. Here is the complete list from the combined efforts of Pappus and Nicomachus.

	Formula	Equivalent
4	$\dfrac{a-b}{b-c} = \dfrac{c}{a}$	$\dfrac{a^2+c^2}{a+c} = b$
5	$\dfrac{a-b}{b-c} = \dfrac{c}{b}$	$a = b + c - \dfrac{c^2}{b}$
6	$\dfrac{a-b}{b-c} = \dfrac{b}{a}$	$c = a + b - \dfrac{a^2}{b}$
7	$\dfrac{a-c}{b-c} = \dfrac{a}{c}$	$c^2 = 2ac - ab$
8	$\dfrac{a-c}{a-b} = \dfrac{a}{c}$	$a^2 + c^2 = a(b+c)$
9	$\dfrac{a-c}{b-c} = \dfrac{b}{c}$	$b^2 + c^2 = c(a+b)$
10	$\dfrac{a-c}{a-b} = \dfrac{b}{c}$	$ac - c^2 = ab - b^2$
11	$\dfrac{a-c}{a-b} = \dfrac{a}{b}$	$a^2 = 2ab - bc$

The most basic fact about these proportions or means is that if $a > c$, then $a > b > c$. (The exception is 10, where b must be selected depending on the relative magnitudes of a and c, and in one of the cases $b = c$.) What is very well known is the following relationship between the first three means. Denote by b_a, b_g, and b_h the arithmetic, geometric, and harmonic means respectively. Then

$$b_a > b_g > b_h \tag{1}$$

The proofs are basic. In all of the statements below equality occurs if and only if $a = c$. First we know that since $(\alpha - \gamma)^2 \geq 0$, it follows that $\alpha^2 + \gamma^2 \geq 2\alpha\gamma$. Apply this to $\alpha = \sqrt{a}$ and $\beta = \sqrt{b}$ to conclude that $a + c > 2\sqrt{ac}$, or $b_a \geq b_g$. Next, we note that $b_h = 2\frac{ac}{a+b}$ or $b_g^2 = b_h b_a$. Thus $b_a \geq b_g \geq b_h$.

What is not quite as well known is that the fourth mean, sometimes called the *subcontrary to the harmonic* mean is larger than all the others except the seventh and the ninth, where there is no greater than or less than comparison over the full range of a and c. The proof that this mean is greater than b_a is again straight forward. We easily see that

$$\begin{aligned} b = \frac{a^2 + c^2}{a + c} &= \frac{(a + c)^2 - 2ac}{a + c} \\ &= 2b_a - \frac{b_g^2}{b_a} \geq b_a \end{aligned}$$

by (1). The other proofs are omitted.

Notice that the first six of the proportions above are all of a specific generic type, namely having the form $\frac{a-b}{b-c} = \cdots$. It turns out that each of the means (the solution for b) are comparable. The case with the remaining five proportions is very much different. Few comparisons are evident, and none of the proportions are much in use today. The chart of comparison of all the means below shows a plus (minus) if the mean corresponding to the left column is greater (less) than that of the top row. If there is no comparison in the greater or less than sense, the word “No” is inserted.

Comparing Pythagorean Proportions

i/j	1	2	3	4	5	6	7	8	9	10	11
1		+	+	-	-	-	No	No	No	No	+
2	-		+	-	-	-	No	No	-	No	No
3	-	-		-	-	-	-	No	-	No	No
4	+	+	+		+	+	No	+	No	+	+
5	+	+	+	-		+	No	+	No	+	+
6	+	+	+	-	-		No	No	No	No	+
7	No	No	+	No	No	No		No	-	No	No
8	No	No	No	-	-	No	No		No	+	+
9	No	+	+	No	No	No	+	No		No	No
10	No	No	No	-	-	No	No	-	No		No
11	-	No	No	-	-	-	No	-	No	No	

Linking qualitative or subjective terms with mathematical proportions, the Pythagoreans called the proportion

$$b_a : b_g = b_g : b_h$$

the **perfect** proportion. The proportion

$$a : b_a = b_h : c$$

was called the **musical** proportion.

8 The Discovery of Incommensurables

Irrationals have variously been attributed to Pythagoras or to the Pythagoreans as has their study. Here, again, the record is poor, with much of it in the account by Proclus in the 4^{th} century CE. The discovery is sometimes given to **Hippasus of Metapontum** (5^{th} cent BCE). One account gives that the Pythagoreans were at sea at the time and when Hippasus produced (or made public) an element which denied virtually all of Pythagorean doctrine, he was thrown overboard. However, later evidence indicates that Theaetetus[23] of Athens (c. 415 - c. 369 BCE)

[23] the teacher of Plato

discovered the irrationality of $\sqrt{3}$, $\sqrt{5}$, ..., $\sqrt{17}$, and the dates suggest that the Pythagoreans could not have been in possession of any sort of "theory" of irrationals. More likely, the Pythagoreans had noticed their existence. Note that the discovery itself must have sent a shock to the foundations of their philosophy as revealed through their dictum *All is Number*, and some considerable recovery time can easily be surmised.

Theorem. $\sqrt{2}$ is incommensurable with 1.

Proof. Suppose that $\sqrt{2} = \frac{a}{b}$, with no common factors. Then

$$2 = \frac{a^2}{b^2}$$

or

$$a^2 = 2b^2.$$

Thus[24] $2 \mid a^2$, and hence $2 \mid a$. So, $a = 2c$ and it follows that

$$2c^2 = b^2,$$

whence by the same reasoning yields that $2 \mid b$. This is a contradiction. ■

Is this the actual proof known to the Pythagoreans? Note: Unlike the Babylonians or Egyptians, the Pythagoreans recognized that this class of numbers was wholly different from the rationals.

"Properly speaking, we may date the very beginnings of "theoretical" mathematics to the first proof of irrationality, for in "practical" (or applied) mathematics there can exist no irrational numbers."[25] Here a problem arose that is analogous to the one whose solution initiated theoretical natural science: it was necessary to ascertain something that

[24] The expression $m \mid n$ where m and n are integers means that m divides n without remainder.

[25] I. M. Iaglom, Matematiceskie struktury i matematiceskoie modelirovanie. [Mathematical Structures and Mathematical Modeling] (Moscow: Nauka, 1980), p. 24.

it was absolutely impossible to observe (in this case, the incommensurability of a square's diagonal with its side).

The discovery of incommensurability was attended by the introduction of indirect proof and, apparently in this connection, by the development of the definitional system of mathematics.[26] In general, the proof of irrationality promoted a stricter approach to geometry, for it showed that the evident and the trustworthy do not necessarily coincide.

9 Other Pythagorean Contributions.

The Pythagoreans made many contributions that cannot be described in detail here. We note a few of them without commentary.

First of all, connecting the concepts of proportionality and relative prime numbers, the theorem of Archytas of Tarantum (c. 428 - c. 327 BCE) is not entirely obvious. It states that there is no mean proportional between successive integers. Stated this way, the result is less familiar than using modern terms.

Theorem. (Archytas) For any integer n, there are no integral solutions a to

$$\frac{A}{a} = \frac{a}{B}$$

where A and B are in the ratio $n : n + 1$.

Proof. The proof in Euclid is a little cumbersome, but in modern notation it translates into this: Let C and D be the smallest numbers in the same ratio as A and B. That is C and D are relatively prime. Let $D = C + E$ Then

$$\frac{C}{D} = \frac{C}{C+E} = \frac{n}{n+1}$$

which implies that $Cn + C = Cn + En$. Canceling the terms Cn, we see that E divides C. Therefore C and D are not relatively prime, a contradiction. ■

[26] A. Szabo "Wie ist die Mathematik zu einer deduktiven Wissenschaft geworden?", Acta Antiqua, 4 (1956), p. 130.

The Pythagoreans also demonstrated solutions to special types of linear systems. For instance, the *bloom* of Thymaridas (c. 350 BCE) was a rule for solving the following system.

$$\begin{aligned} x + x_1 + x_2 + \ldots + x_n &= s \\ x + x_1 &= a_1 \\ x + x_2 &= a_2 \\ \ldots & \\ x + x_n &= a_n \end{aligned}$$

This solution is easily determined as

$$x = \frac{(a_1 + \ldots + a_n) - s}{n - 2}$$

It was used to solve linear systems as well as to solve indeterminate linear equations.

The Pythagoreans also brought to Greece the earth-centered cosmology that became the accepted model until the time of Copernicus more that two millenia later. Without doubt, this knowledge originated in Egypt and Babylon. Later on, we will discuss this topic and its mathematics in more detail.

References

1. Russell, Bertrand, *A History of Western Philosophy*, Simon and Schuster Touchstone Books, New York, 1945.